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PORTFOLIO SELECTION USING POWER LAW WITH EXPONENTIAL CUT-OFF UTILITY FUNCTION

Abstract. *This paper considers the optimal portfolio selection problem in a multiple period setting where the investor maximizes the expected utility of the terminal wealth based on the Power law with exponential cut-off (PLEC) utility function, in a stochastic market. Optimal portfolio selection in a multi-period setting where risk preferences are allowed to change in every period is discussed. The change in the market conditions is modelled according to a Markov Chain. Dynamic programming approach is used to obtain the solution of the optimal policy and the corresponding value function is applied. The applicability of PLEC utility is demonstrated here, in order to obtain an optimal policy which depends on the wealth invested and also on the period of investment along with the state of the market. The focus is on using the different PLEC utility functions for representing various investors of the market.*

Keywords*: Portfolio optimization, Markov chain, Dynamic Programming, PLEC utility, risk preferences.*

JEL Classification: B41, C61, G11

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1. Introduction

All financial investments involve complex decision making criteria within themselves. The decision, as to how the given wealth for investment will be distributed among the various available financial instruments, is a very tough one. Every investor wants to make his portfolio an optimum one, taking into consideration the market conditions and his risk preference. The manner in which one can arrive at an optimum decision differs from scenario to scenario. Different investors may have different end goals and different preferences towards financial instruments. Their decision also depends on their wealth and time. In order to achieve these goals various methods have been developed. In the pioneering work by Markowitz (1952), minimizing the variance of the returns of the assets, in order to obtain the best portfolio optimization decision, was suggested. However it suffers with a few drawbacks particularly of not taking the investors' time horizon into consideration. Merton (1969) and Merton (1971) have corrected some of his errors.

Classical method of optimization depends only on mean and variances of returns; however, this does not take into account investors preferences and the amount of wealth to be invested. Optimization of a portfolio based on utility functions of investors will be rather more appropriate and has been considered recently. A utility function is non-decreasing, real valued function defined on real numbers. Since Bernoulli (1954) has given importance to the use of utility functions as a basis for investment decisions, their use have increased tremendously. In the recent years a lot of new forms of utility functions have been developed which can exhibit one or more risk preferences within its form, for various parameter values. Although constant relative risk aversion (CRRA) is the most widely used risk preference structure, its use in the real world is often limited as discussed by Xie(2000). Similarly, it has been argued by Pratt (1964) that utility functions exhibiting decreasing absolute risk aversion (DARA) are most appropriate to model people's preferences. However there are very less empirical evidences on this. Similar conclusions are reached in the past when people have tried to claim that sticking to one kind of preference structure is "better" to model investor behaviour. Hence, one requires a utility function which is not limited in use and can represent a larger section of investors with varied and mixed risk attitudes.

Some of the commonly used utility functions are quadratic utility, power utility, logarithmic utility and exponential utility as highlighted by Sharpe (2007). Among the new lot of utility functions the most prominent ones are the Expo-power utility by Saha (1993), which can exhibit all forms of Absolute risk aversion (ARA) and the Relative risk aversion (RRA) measures, Power risk aversion (PRA) utility by Xie (2000), which can exhibit Constant absolute risk aversion (CARA) and CRRA, A new utility function called the PLEC utility is suggested by Arif and Pakkala (2014), which incorporates all most all the risk preference structures of investors as described by Pratt (1964) and Arrow (1965).

Many have worked with utility functions in order to obtain an optimal solution to the portfolio problem. Mossin (1968) has examined utility functions leading to myopic policies; Merton (1969) considered special utility functions with logarithmic and power structures. In a survey paper by Steinbach (2001), 208 papers, which show the diversity of different models and approaches used to analyze this problem of optimum portfolio based on utility functions, for both single period and multi-period cases, are reviewed. As highlighted by Çanakoğlu and Özekici (2010) a huge amount of work is done in this direction by a number of academicians.

In a multi-period setting, as highlighted by Çelikyurt and Özekici(2007), Çanakoğlu and Özekici (2009) and Çanakoğlu and Özekici (2010), the assumption of uncorrelated asset returns is not realistic. One may achieve this dependence or correlation through a stochastic market process, like a Markov chain, which affects all deterministic and probabilistic parameters of the model. Solution of these models is given through dynamic programming techniques.

Here we show that the PLEC utility can be used to model various possibilities of the real world. It is also shown that an optimum solution for the portfolio problem exists and is obtained through dynamic programming for a multi period set up. We have chosen this utility in particular because of the fact that it can exhibit all the risk preference structures of different types of investors. This way we can remove any bias occurring in the solution, due to the use of different utility functions.

Section 2 describes the basic notations and assumptions on the market in this study. Section 3 gives a brief introduction on the PLEC utility function. In section 4an example is presented through which, sections of the market utility function is represented through different PLEC utility functions. In section 5, the various models that can be incorporated using the PLEC utility function are studied and also it is shown that the optimum solution for these models exists. In Section 6, the solution of the dynamic problem equation for the different models mentioned above is worked out. Section 7 contains illustrations based on the above models solved through dynamic programming technique. Section 8 consists of the conclusions.

2. Model description and assumptions on the market

The returns of the assets of a portfolio change continuously in a random fashion. Distributions such as normal and lognormal have been used to model them. The variation occurring in them may be due to internal as well as external factors. These factors could be at a local and/or at a global level. Investment decisions are affected by these factors as well as the correlation among asset returns. Modelling a stochastic financial market by a Markov chain is a reasonable approach in this situation as highlighted by Çanakoğlu and Özekici (2009). They argue that recently there is growing interest in the literature, to use a stochastic market process, in order to modulate various parameters of the financial model, hence making it more realistic. They also point out that continuous-time Markov chains with a discrete state space are used in a number of papers including, for example, Bäuerle and Rieder (2004) and Yin and Zhou (2004), to modulate model parameters in portfolio selection and stock trading problems. Hernández-Hernández and Marcus (1999), Bielecki et al. (1999) and Nagai and Peng (2002) are a few others who have done similar work in this area.

2.1. Wealth Dynamics Equation

Suppose that the state of the market in period *n* is denoted by Y_n and $Y = \{Y_n\ ; n = 0,1,2, \dots\}$ is a Markov chain with a discrete state space, E and transition matrix, Q. Let R(i) denote the random vector of asset returns in any period given that the stochastic market is in state *i*. The means, variances and covariance of asset returns depend only on the current state of the stochastic market. The market consists of one riskless asset with known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$; and m risky assets with random returns $R^n(i) = \left(R_1^n(i), R_2^n(i), \dots, R_m^n(i)\right)$ in period n if the state of the market is *i*. We assume that the random returns in consecutive periods are conditionally independent given the market states. In other words, $R^n(i)$ is independent of $R^k(j)$ for all periods $k \neq n$ and states *i* and *j*. Moreover, $R^n(i)$ and $R^k(i)$ are independent and identically distributed random vectors whenever $k \neq n$ This implies that the distributions of the asset returns depend only on the state of the market independent of time. For this reason, we will let $R(i) = R^n(i)$ denote the random return vector in any period *n* to simplify the notations here.

Let $r_k(i) = E[R_k(i)]$ denote the mean return of the k^{th} asset in state *i* and $\sigma_{kj}(i) = Cov\left(R_k(i), R_j(i)\right)$ denote the covariance between k^{th} and j^{th} asset returns in state *i*. The excess return of the k^{th} asset in state *i* is $R_k^e(i) = R_k(i) - r_f(i)$. It follows that,

$$r_k^e(i) = E[R_k^e(i)] = r_k(i) - r_f(i)$$

$$\sigma_{kj}(i) = Cov\left(R_k^e(i), R_j^e(i)\right)$$

The notation here is such that $r_f(i)$ is a scalar and $r(i) = \left(r_i(i), r_2(i), \dots, r_m(i)\right)$ and $r^e(i) = \left(r_1^e(i), r_2^e(i), \dots, r_m^e(i)\right)$ are column vectors for all *i*.

Let X_n denote the amount of investor's wealth at period n and let the vector $u = \left(u_1, u_2, \dots, u_{m,}\right)$ denote the amounts invested in risky assets (1, 2, ..., *m*) respectively. Given any investment policy, the stochastic evolution of the investor's wealth follows the so-called wealth dynamics equation,

$$\begin{aligned} X_{n+1}(u) &= R(Y_n)'u + \left(X_n - 1'u\right)r_f(Y_n) \\ &= r_f(Y_n)X_n + R^e(Y_n)'u \end{aligned}$$

where 1' = (1,1,...,1) is the column vector consisting of 'm' ones. In a multi-period set-up, the wealth is reinvested recursively in each period using this equation.

The assumptions regarding the model formulation can be summarized as follows: (a) There is unlimited borrowing and lending at the prevailing return of the riskless asset in any period, (b) Short selling is allowed for all assets in all periods, (c) No capital additions or withdrawals are allowed throughout the investment horizon, and (d) Transaction costs and fees are negligible.

3. Power Law with Exponential Cut off (PLEC) utility function

A utility function U is a non-decreasing real valued function defined on real numbers. An individual may have various risk attitudes, which can be studied based on the first derivative of the utility function. The shape of the utility function also gives a notion of what exactly the investor puts in the risk concept. A risk-averse utility will be convex; a risk seeking utility will be concave, whereas a risk neutral utility will be linear as described in Pratt (1964).

Pratt (1964) and Arrow (1965) suggest the ratio

$$\mathrm{A(x)} = -\mathrm{U''(x)}/\mathrm{U'(x)}$$

as a measure of absolute risk aversion (ARA). ARA explains how, within, risk aversion the investor preferences change in wealth. When the first derivative of A(x) i.e., A'(x)<0 the utility exhibits decreasing absolute risk aversion (DARA), when A'(x)=0 it exhibits constant absolute risk aversion (CARA) and when A'(x)>0 it exhibits increasing absolute risk aversion (IARA).

Also, within risk aversion, one can study the changes in investor preferences, with respect to percentage of wealth invested, with the help of the relative risk aversion (RRA) measure which is defined as

$$R(x) = xA(x)$$

When the first derivative of R(x) i.e., R'(x)<0 the utility exhibits decreasing relative risk aversion (DRRA), when R'(x)=0, it exhibits constant relative risk aversion (CRRA) and, when R'(x)>0, it exhibits increasing relative risk aversion (IRRA).

Arif and Pakkala (2014) have introduced a new utility function called as Power Law with Exponential Cut off (PLEC) utility which can exhibit all most all of these risk preference structures. The PLEC utility function is defined as,

$$U(x) = \theta - Ce^{\beta x}x^{\alpha} \tag{3.1}$$

for

$$\alpha + \beta x < 0. \tag{3.2}$$

Here θ is so chosen that it makes the value of the utility (if $\theta=0$, the properties still hold), U, positive for all values of the wealth x > 0. C>0 is the coefficient of the utility function. Here, α and β are the parameters. (3.2) is a consequence of the condition, for any utility function, to have a positive first derivative.

The special cases $\alpha = 0$ gives us the exponential utility function and $\beta = 0$ gives us the power utility function both of which are members of the HARA family.

As described by Arif and Pakkala(2014), PLEC utility can represent a risk liking as well as a risk averse investor. Also within risk aversion it can cover investors who could be increasing absolute risk averse (IARA), decreasing absolute risk averse (DARA), constant absolute risk averse (CARA), increasing relative risk averse (IRRA), decreasing relative risk averse (DRRA) and constant relative risk averse (CRRA).

4. Representation of an investor in the market through PLEC utility function.

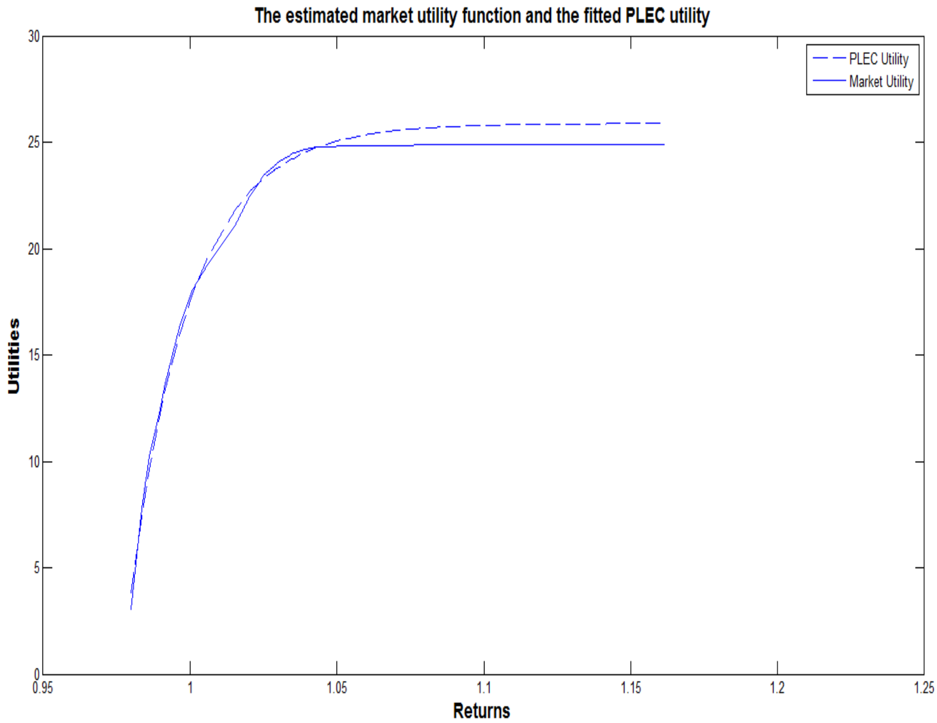
A financial market may contain various types of investors with varying degrees of risk. Once we know how the market utility function behaves, we can check whether the market contains investors having PLEC utility or not.

Rosenberg and Engle (2002) and Detlefsenet. al (2008) have worked in the area of estimating the market utility function based on the empirical pricing kernel. They explain how, based on a market reflecting index, empirical pricing kernels can be estimated. The estimation procedure is based on historical and risk neutral densities and these distributions are derived with stochastic volatility models that are widely used in industry. Detlefsenet. al(2008)then describes a method to obtain individual investor utility function from the market utility function.

We have used the CNX NIFTY index of the national stock exchange (NSE) of India to estimate the empirical pricing kernel. As described by Detlefsenet. al(2008) the pricing kernel is obtained by estimating the risk neutral and the subjective density and then deriving the pricing kernel. This approach does not impose a strict structure on the kernel. Moreover, we use accepted parametric models because nonparametric techniques for the estimation of second derivatives depend a lot on the bandwidth selection although they yield the same pricing kernel behaviour over a wide range of bandwidths. For the risk neutral density we use Heston’s stochastic volatility model that is popular both in academia and in industry. The historical density is more difficult to estimate because the drift is not fixed. Hence, the estimation depends more on the model and the length of the historical time series. The GARCH model is used for the historical density, which is the discrete version of the continuous model for the risk neutral density.

To the market utility function we fit a PLEC utility function and we see that some portion of the market utility behaves in a PLEC manner. This implies that there is an investor(s) who have utilities behaving according to a PLEC utility function. The Figure 1 below shows the market and fitted PLEC utility functions.

Figure 1.The estimated market utilities and the fitted PLEC utilities.



The values of the parameters is obtained as θ=25.8815, α=-49.4953 and β=2.1265.

As described by Clausetet. al. (2009) the PLEC function is good in describing behaviour at extreme ends of the distributions. A similar observation is seen in Figure 1 as well. The investors in the lower end of wealth are very well represented with the help of the PLEC utility function with the parameters mentioned above. Similarly other sections of the market utility can be represented through other PLEC utility functions. For example an inward bend can be observed in the market utility which indicated the presence of risk liking investors in the market this section can very well be represented again with PLEC utility with suitable parameters.

Thus this shows that there is an investor in the market whose utility behaviour can very well be modelled through a PLEC utility function. Hence investors with PLEC utilities exist in the market and can very well be modelled with the PLEC function proposed by Arif and Pakkala (2014). This justifies the use of PLEC utility in our study. Moreover we recommend the use of PLEC utility because of its unique ability to represent risk liking as well as risk averse investors as mention in section 3.

5. General distribution model and existence of optimum solution

In this study it is assumed that the distributions of the risky asset returns are arbitrary. The analysis in this section contains the proof of the existence of optimum solution for the PLEC utility described above. Then two special cases are mentioned where one of the parameters is taken to be zero, which gives the exponential and power model of utility function. The characteristics of the optimal investment policy for *m*=1 risky asset is also discussed here.

It is known that $g(i,x,u)$ is the expected utility using the investment policy *u* given that the market is in state *i* and the amount of money available for investment is *x*. Hence

$$g(i,x,u) = \sum_{j=1}^{s} Q(i,j)\, E\left[U\left(j, r_f x + R^e(i)'u\right)\right]$$

$$= \sum_{j=1}^{s} Q(i,j)\theta_j - \sum_{j=1}^{s} Q(i,j)C(j)\, e^{\beta x r_f} E\left[e^{\beta R^e(i)'u}\left(x r_f + R^e(i)'u\right)^{\alpha}\right]$$

where$\theta_j$and $C(j)$are positive constants.

$$g(i,x,u) = K(i) - C'(i) e^{\beta x r_f} E\left[e^{\beta R^e(i)'u}\left(x r_f + R^e(i)'u\right)^{\alpha}\right] \qquad (5.1)$$

where $K(i) = \sum_{j=1}^{s} Q(i,j)\theta_j > 0$ and $C'(i) = \sum_{j=1}^{s} Q(i,j)C(j) > 0$.

It can be easily seen that since every function inside the summation in (5.1) is monotone,$g(i,x,u)$is monotone in *u* for all *x* and *i*. Therefore to find the optimum portfolio of risky assets, it is enough if we set the gradient to zero so that the optimality condition becomes

$$\nabla_k g(i,x,u) = \frac{\delta g(i,x,u)}{\delta u_k}$$

$$\nabla_k g(i,x,u) = -C'(i)e^{\beta x r_f} E\left[e^{\beta R^e(i)'u} R_k^e(i)\left(xr_f + R^e(i)'u\right)^\alpha \left[\beta + \frac{\alpha}{xr_f + R^e(i)'u}\right]\right] = 0 \qquad (5.2)$$

for all k.

Define $A^k(i,x,u) = -C'(i)e^{\beta x r_f} E\left[e^{\beta R^e(i)'u} R_k^e(i)\left(xr_f + R^e(i)'u\right)^\alpha \left[\beta + \frac{\alpha}{xr_f + R^e(i)'u}\right]\right] = 0$ so that the optimality condition (5.2) can be written for all m assets as,

$$A(i,x,u) = [A^1(i,x,u), A^2(i,x,u), \ldots, A^m(i,x,u)] = [0,0,\ldots,0] \qquad (5.3)$$

Lemma 5.1: *The function $A^k(i,x,u)$ and hence $A(i,x,u)$ are strictly decreasing in u_k for all x, i and k.*

This follows by noting that,

$$\frac{\delta A^k(i,x,u)}{\delta u_k} = -C'(i)e^{\beta x r_f} R_k^e(i) E\left[e^{\beta R^e(i)'u}\left(xr_f + R^e(i)'u\right)^\alpha \left[\beta^2 R_k^e(i) + \frac{2\alpha R_k^e(i)\beta}{xr_f + R^e(i)u} + \frac{\alpha(\alpha-1)R_k^e(i)}{\left(xr_f + R^e(i)'u\right)^2}\right]\right] < 0 \qquad (5.4)$$

for all x, i and k. This follows because of condition (3.2) which holds for any risk-averse investor.

Theorem 5.1: *The optimal policy is the unique continuously differentiable function $u^*(i,x) = [u_1^*(i,x), u_2^*(i,x), \ldots, u_m^*(i,x)]$ that satisfies $A\left(i,x,u^*(i,x)\right) = 0$ for any x and all i.*

Proof:

For the existence of the optimal policy satisfying the optimality condition (5.2), consider the Hessian of the objective function g which is the symmetric matrix,

$$\nabla_{k,l}^2 g(i,x,u) = \frac{\delta \nabla_k g(i,x,u)}{\delta u_l} = \frac{\delta^2 g(i,x,u)}{\delta u_l \delta u_k}$$

$$\nabla^2_{k,l} g(i,x,u) = -C'(i)e^{\beta x r_f} E\left[e^{\beta R^e(i)'u} R^e_k(i)\, R^e_l(i)\left(xr_f + R^e(i)'u\right)^{\alpha-2}\left[(\beta(xr_f + R^e(i)'u) + \alpha)^2 - \alpha\right]\right]$$
(5.5)

For any vector $z = [z_1, z_2, \dots, z_m]$,

$$z'\left(\nabla^2 g(i,x,u)\right)z$$
$$= -C'(i)e^{\beta x r_f} E\left[e^{\beta R^e(i)'u}\left(xr_f + R^e(i)'u\right)^{\alpha-2}\left[(\beta(xr_f + R^e(i)'u) + \alpha)^2 - \alpha\right]\left(\sum_{k=1}^{m} z_k R^e_k(i)\right)^2\right] < 0$$

Implying that the Hessian of $g(i,x,.)$ is negative definite for all i and x. Every negative definite matrix has an inverse which is also negative definite. The implicit function theorem can be applied to the optimality condition (5.2) which can be rewritten as

$$\nabla g(i,x,u) = [\nabla_1 g(i,x,u), \nabla_2 g(i,x,u), \dots, \nabla_{\mathrm{m}} g(i,x,u)] = [0,0,\dots,0].$$

Note that $\nabla g(i,x,u)$ is a continuously differentiable function. The Hessian in (5.4) gives the matrix of the first order derivatives of $\nabla g(i,x,u)$ with respect to $u = [u_1, u_2, \dots, u_m]$. This is clearly invertible at any fixed point (i,x,u), since itis negative definite. The proof is now immediate through the implicit function theorem for $\nabla g(i,x,u) = A(i,x,u) = 0$.

Therefore the existence and uniqueness of the optimal policy or portfolio u^* is established, where $u^*_k(x)$ is the amount of money invested in asset *k* if the wealth of the investor is *x*.

The structure of the three special cases of interest, that is considered here, will be seen in the following sub-sections.

5.1.Constant risk tolerance model

If the risk tolerance of the investor is constant so that $\alpha = 0$, then theoptimality condition (5.2) becomes $E\left[R^e_k \beta e^{\beta R^{e\prime} u}\right] = 0$ for all *k*. Note that for $\beta = 0$ a trivial solution is obtained. Therefore $\beta \neq 0$ is to be considered.

5.2.Decreasing absolute risk tolerance model

If the risk tolerance of the investor is decreasing so that $\beta = 0$, then the optimality condition (5.2) becomes $E\left[R^e_k \alpha\left(xr_f + R^{e\prime}u\right)^{\alpha-1}\right] = 0$ for all *x* and *k*. Note that for $\alpha = 0$ a trivial solution is obtained. Therefore $\alpha \neq 0$ is to be considered.

5.3. Risk tolerance with one risky and one risk-free asset

For this case, since the distribution of the risky asset is general, one needs to consider the generalized optimality condition (5.2).It is known that $\mathrm{A(i, x, u)}$is decreasing in u. Note that

$$A^k(i,x,0) = -C'(i)e^{\beta x r_f}\ E\left[R_k^e(i)\left(xr_f\right)^{\alpha-1}\left[\beta x r_f + \alpha\right]\right]$$
$$= -C'(i)e^{\beta x r_f}\left(xr_f\right)^{\alpha-1}\left[\beta x r_f + \alpha\right]\mu$$

where $\mu = \bar{r} - r_f$ is the mean of the excess return. Therefore, the sign of $A^k(i,x,0)$depends on the sign ofμ.

Some of the characteristics on the structure of the optimum policy can easily be seen as follows. If$\mu > 0$, i.e., the mean return of the risky asset exceeds that of the risk free asset then,$A^k(i,x,0) > 0$ so that $u^*(i,x) > 0$ since $A^k(i,x,u)$is strictly decreasing in u and the optimal decision satisfying $A^k\left(i,x,u^*(x)\right) = 0$, is greater than zero for all x. This indicates that some positive amount of current wealth is invested in the risky asset. Moreover, when$\mu < 0$ we get,$u^*(i,x) < 0$,for all x by a similar argument. This indicates that since the return of risk free asset exceeds mean return of the risky asset the policy is to sell short the risky asset and to invest $x - u^*(i,x) > x$ in the risk free asset.

6. Dynamic programming solution based on the PLEC utility

Now the dynamic programming equation and optimum portfolio, is obtained for an investor with varied risk preferences under the model described above.

6.1. Individual risk tolerance model represented by PLEC utility function

The optimum solution for the portfolio problem for an investor under this category can be summarised in Theorem 6.1 as:

Theorem 6.1: *Let the utility function of the investor be the PLEC function given by*

$$U(i,x) = K(i) - C(i)(x-\eta)^{\alpha}e^{\beta x}$$

for some $\eta > 0$ *and suppose that the risk-less asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (5.5) is*

$$v_n(i,x) = K_n(i) - C_n(i)(x-\eta_{n+1})^{\alpha}e^{x\beta_{n+1}}$$

and the optimal portfolio is,

$$u_n^*(i,x) = \gamma_n(i,x)(xr_f - \eta_{n+1}) \qquad (6.1)$$

where

$$\eta_n = \eta_T r_f^{T-n},\ \beta_n = \frac{\beta}{r_f^{T-n}}\ K_{n-1} = Q^{T-n}K\ ,C_n = \hat{Q}_\alpha^{\ T-n}C \qquad (6.2)$$

and

$$\hat{Q}_\alpha(i,j) = r_f^\alpha E\left[e^{\beta_{n+1}R^e(i)\gamma_n(i,x)}\left(r_f + R^e(i)'\gamma_n(i,x)\right)^\alpha\right]Q(i,j)$$

for n=0,1,...,T-1 where $\gamma_n(i,x)$ *satisfies,*

$$E\left[R_k^e(i)(R^e(i)'\gamma_n(i,x)+1)^{\alpha-1}e^{\beta_{n+1}(x-\eta_{n+1})R^e(i)'\gamma_n(i,x)}\left(\alpha+\beta_{n+1}(x-\eta_{n+1})\left(1+R^e(i)'\gamma(i,x)\right)\right)\right] = 0 \qquad (6.3)$$

for all assets k=1,2,...,m, all states i and all periods n.

Proof: The induction starting with the boundary condition is used as,

$$v_T(i,x) = K(i) - C(i)(x-\eta)^\alpha e^{x\beta}$$

and

$$g_{T-1}(i,x,u) = \sum_{j=1}^{s} Q(i,j)\, E\left[U\left(j, r_f x + R^{e'}(i)u\right)\right]$$

$$= \sum_{j=1}^{s} Q(i,j)\left[K(j) - C(j)E\left[e^{\beta(xr_f + R^e(i)'u)}\left(xr_f + R^e(i)'u - \eta\right)^\alpha\right]\right]$$

$$= K'(i) - C'(i)E\left[e^{\beta(xr_f + R^e(i)'u)}\left(xr_f + R^e(i)'u - \eta\right)^\alpha\right]$$

is obtained, where $K'^{(i)} = \sum_{j=1}^{s} Q(i,j)K(i)$ and $C'(i) = \sum_{j=1}^{s} Q(i,j)C(i)$

Let u^* be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

One can see that the objection function $g_{T-1}(i,x,u)$ is in the form of a PLEC utility function. The assumption on the existence of an optimum solution implies that the optimal policy can be found using the first order condition,

$$E\left[e^{\beta\left(xr_f+R^e(i)'u^*_{T-1}(i,x)\right)}\left(xr_f+R^e(i)'u^*_{T-1}(i,x)-\eta\right)^{\alpha}R^e_k(i)\left[\beta+\frac{\alpha}{xr_f+R^e(i)'u^*_{T-1}(i,x)-\eta}\right]\right]=0$$

for all *k=1,...,m*. Defining the vector function $\gamma(i,x)=(\gamma^1(i,x),\gamma^2(i,x),\dots,\gamma^m(i,x))$ such that $\gamma(i,x)=u^*(i,x)/(xr_f-\eta)$one obtains$u^*(i,x)=\gamma(i,x)(xr_f-\eta)$ so that the optimality condition can be re-written as,

$$E\left[e^{\beta(xr_f+R^e(i)'\gamma(i,x)(xr_f-\eta))}\left(xr_f+R^e(i)'\gamma(i,x)(xr_f-\eta)-\eta\right)^{\alpha}R^e_k(i)\left[\beta+\frac{\alpha}{xr_f+R^e(i)'\gamma(i,x)(xr_f-\eta)-\eta}\right]\right]=0$$

Or

$$E\left[e^{\beta\left(R^e(i)'\gamma(i,x)(xr_f-\eta)\right)}\left(1+R^e(i)'\gamma(i,x)\right)^{\alpha-1}R^e_k(i)\left[\beta(xr_f+R^e(i)'\gamma(i,x)\left(xr_f-\eta\right)-\eta)+\alpha\right]\right]=0$$

The value function at time *T-1* is re-written for the optimal policy as,

$$\begin{aligned}
v_{T-1}(i,x)&\\
&=K'(i)-C'(i)E\left[e^{\beta(xr_f+R^e(i)'\gamma(i,x)(xr_f-\eta))}\left(xr_f+R^e(i)'\gamma(i,x)\left(xr_f-\eta\right)-\eta\right)^{\alpha}\right]\\
&=K'(i)-C'(i)E\left[(xr_f-\eta)^{\alpha}e^{\beta(xr_f+R^e(i)'\gamma(i,x)(xr_f-\eta))}\left(1+R^e(i)'\gamma(i,x)\right)^{\alpha}\right]\\
&=K'(i)-\hat{Q}_{\alpha}C'(i)(x-\eta/r_f)^{\alpha}e^{\beta xr_f}\\
&=K'(i)-\hat{Q}_{\alpha}C'(i)(x-\eta/r_f)^{\alpha}e^{\beta xr_f}\\
&=K_{T-1}(i)-C_{T-1}(i)(x-\eta_{T-1})^{\alpha}e^{\beta_{T-1}x}
\end{aligned}$$

and the value function is still PLEC like the utility function. This follows from noting that$K_{T-1}=K'$, $\eta_{T-1}=\eta/r_f$, $\beta_{T-1}=\beta r_f$and $C_{T-1}=\hat{Q}_{\alpha}C'(i)$.This completes the proof for *n=T-1*.

Suppose now that the induction hypothesis holds for periods T, *T-1*, *T-2,...,n*. Then for period *n-1*,

$$g_{n-1}(i,x,u) = \sum_{j=1}^{s} Q(i,j)\, E\left[v_n\left(j, r_f x + R^e(i)'u\right)\right]$$

$$g_{n-1}(i,x,u) = K'(i) - C'(i)E\left[e^{\beta(xr_f+R^{e'}(i)u)}\left(xr_f + R^e(i)'u - \eta\right)^{\alpha}\right] \qquad (6.4)$$

Once again the objective function is of the form (5.1). The assumption regarding the existence of optimal solution implies that the optimal policy can be found using the first order condition

$$E\left[e^{\beta_n\left(xr_f+R^e(i)'u^*_{n-1}(i,x)\right)}\left(xr_f + R^e(i)'u^*_{n-1}(i,x) - \eta_n\right)^{\alpha} R^e_k(i)\left[\beta_n + \frac{\alpha}{xr_f + R^e(i)u^*_{n-1}(i,x) - \eta_n}\right]\right] = 0$$

and letting $\gamma_{n-1}(i,x) = u^*_{n-1}(i,x)/(xr_f - \eta_n)$one obtains the optimum policy as $u^*_{n-1}(i,x) = \gamma_{n-1}(i,x)(xr_f - \eta_n)$, where $\gamma_{n-1}(i,x)$ satisfies

$$E\left[e^{\beta_n(xr_f+R^e(i)'\gamma_{n-1}(i,x)(xr_f-\eta_n))}\left(xr_f + R^e(i)'\gamma_{n-1}(i,x)(xr_f - \eta_n) - \eta\right)^{\alpha} R^e_k(i)\left[\beta_n + \frac{\alpha}{xr_f + R^e(i)\gamma_{n-1}(i,x)(xr_f - \eta_n) - \eta}\right]\right] = 0$$

For *k=1,2,...,m*. If one inserts the optimal policy in the value function using (6.4) it can be seen that,

$$\begin{aligned}
v_{n-1}(i,x) &= K'_n(i) - C'_n(i)E\left[e^{\beta_n(xr_f+R^e(i)'\gamma_{n-1}(i,x)(xr_f-\eta_n))}\left(xr_f + R^e(i)'\gamma_{n-1}(i,x)\left(xr_f - \eta_n\right) - \eta_n\right)^{\alpha}\right] \\
&= K'_n(i) - C'_n(i)E\left[\left(xr_f - \eta_n\right)^{\alpha} e^{\beta_n(xr_f+R^e(i)'\gamma_{n-1}(i,x)(xr_f-\eta_n))}\left(1 + R^e(i)'\gamma_{n-1}(i,x)\right)^{\alpha}\right] \\
&= K'_n(i) - \hat{Q}_\alpha C'_n(i)(x - \eta_n/r_f)^{\alpha} e^{\beta_n r_f} \\
&= K_{n-1}(i) - C_{n-1}(i)(x - \eta_{n-1})^{\alpha} e^{\beta_{n-1} x}
\end{aligned}$$

and the value function is still PLEC. Note that the recursions $K_{n-1} = K'_n \eta_{n-1} = \eta_n / r_f$, $\beta_{n-1} = \beta_n r_f$ and $C_{n-1} = \hat{Q}_\alpha C'_n$ with boundary values $C_T = C$,$\eta_T = \eta$ and $\beta_T = \beta$ give the explicit solutions in (6.2). This completes the proof.

The solution of exponential (α=0) and power(β=0) model is a special case of this model which is already been stated and proved by Çanakoğlu and Özekici(2009).

From the model solution we see that the optimum policy depends on the state of the market, period of investment and wealth of the investor. The model solution also depends on the joint distribution of the risky asset returns as prescribed by (6.3) in a given market state, irrespective of future expectations on the stochastic market. Another important feature is the dependence of the optimal policy on the period of investment. This condition makes sure that with every changing period the policy itself could be different. This model can incorporate more risk preference structures that investors can exhibit with the change in state of the market and their wealth.

6.2. PLEC utility function with η=0

Theorem 6.2: *Let the utility function of the investor be the PLEC function given by*

$$U(i,x) = K(i) - C(i)(x)^\alpha e^{\beta x}$$

the optimal solution of the dynamic programming equation (5.5) is

$$v_n(i,x) = K_n(i) - C_n(i)(x)^\alpha e^{x\beta_{n+1}}$$

and the optimal portfolio is,

$$u_n^*(i,x) = \gamma_n(i,x) x r_f(i) \tag{6.5}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}\ K_{n-1} = Q^{T-n}K\ ,\ C_n = \hat{Q}_\alpha{}^{T-n} C \textit{ and} \tag{6.6}$$

$$\hat{Q}_\alpha(i,j) = r_f^\alpha(i) E\left[e^{\beta_{n+1} R^e(i)\gamma_n(i,x)} \left(1 + R^e(i)'\gamma_n(i,x)\right)^\alpha\right] Q(i,j)$$

for n=0,1,...,T-1 where $\gamma_n(i,x)$ satisfies,

$$(6.7) E\left[R_k^e(i)(R^e(i)'\gamma_n(i,x)+1)^{\alpha-1} e^{\beta_{n+1} x R^e(i)'\gamma_n(i,x)} \left(\alpha + \beta_{n+1} x\left(1 + R^e(i)'\gamma_n(i,x)\right)\right)\right] = 0$$

for all assets k=1,2,...,m and all i.

Proof: The proof is similar to the proof of Theorem 6.1 and will not be repeated here.

In this special case η=0, it is clear that the optimal policy in (6.5) is not myopic since there is dependence on n. This property is unlike other utility functions considered by Çanakoğlu andÖzekici (2009, 2010) At any time *n*, the total amount of money invested in the risky assets depends on the market state *i* and wealth x as well. Since the total risky investment is $1'u_n^*(i,x) = 1'\gamma_n(i,x)xr_f(i)$ it follows that $xr_f(i)\sum_{k=1}^{m}\gamma_n^k(i,x)$ is the portion of total wealth that is invested in the risky assets if the market is in state *i* and the period is *n*. Also observe that for PLEC utility, the composition of the portfolio depends on the market state, investment period as well as the wealth.

If stock prices are dependent on the previous day's price, then it follows a Markov chain. When stock prices depend on the previous two days it does not follow a Markov chain. However by redefining the state space we can modify the process as a Markov chain. Similarly when the stock prices depend on the previous three days the prices can be modelled as a Markov Chain by extending and redefining the states. Hence, Markov chain can be used to model the stock prices in many situations. In the following example it is assumed that the price depends on previous two days.

7. Illustrations

In this section, computational issues are addressed and it is demonstrated how the results obtained can be put to work, through a numerical illustration, for the PLEC utility under the various cases. Consider a market with three risky assets and one riskless asset where the returns of the risky assets follow an arbitrary multivariate distribution. The illustration is based on data obtained during September 2003 to September 2013 from daily return information of three assets (Reliance, ACC and Tata Steel) traded in National stock exchange; and the daily respective treasury rate of RBI. The states of the market are classified by considering whether the S&P CNX Nifty index went up or down during the previous 2 days. Therefore, in order to model prices as a Markov chain, 4 states are defined and labelled as *1=(up, up), 2=(up, down), 3= (down, up)*, and *4= (down, down)* depending on the movement of prices on previous days. The daily interest rates for all states were approximately equal to 1 and our assumption is satisfied. Let the transition matrix be denoted as Q. Some of the transitions are not possible because of the manner in which the states are defined. For example, the transition from state 1 to state 3 is not possible. This is so, because in state 1 the market has gone up twice and in state 3 the market has gone down first and then up. Hence the two states do not overlap, which averts this transition.

Using historical data the transition probability matrix Q of the Markov chain is obtained as,

$$Q = \begin{bmatrix} 0.306 & 0.694 & 0 & 0 \\ 0 & 0 & 0.371 & 0.629 \\ 0.577 & 0.423 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Notice that in the above given transition probability matrix, although theoretically possible, there is no transition from state 4 to state 4.

Table 1.The return from the riskless asset and the expected return of each risky asset for each state

i	r_f	$\mu_1(i)$	$\mu_2(i)$	$\mu_3(i)$
1	1.0002	1.012	1.008	1.011
2	1.0002	0.999	0.999	0.999
3	1.0002	1.004	1.002	1.002
4	1.0002	0.990	0.994	0.991

and the covariance matrices for the four states are

$$\Sigma(1) = \begin{pmatrix} 0.419 & 0.169 & 0.241 \\ 0.169 & 0.358 & 0.177 \\ 0.241 & 0.177 & 0.615 \end{pmatrix} \qquad \Sigma(2) = \begin{pmatrix} 0.269 & 0.097 & 0.169 \\ 0.097 & 0.298 & 0.148 \\ 0.169 & 0.148 & 0.686 \end{pmatrix}$$

$$\Sigma(3) = \begin{pmatrix} 0.298 & 0.107 & 0.169 \\ 0.107 & 0.301 & 0.133 \\ 0.169 & 0.133 & 0.133 \end{pmatrix} \qquad \Sigma(4) = \begin{pmatrix} 0.882 & 0.149 & 0.245 \\ 0.149 & 0.350 & 0.198 \\ 0.245 & 0.198 & 0.646 \end{pmatrix}$$

Note that these values are obtained by multiplying the actual numbers by 1,000 for simplification. We consider the problem of investors with initial wealth *x = Rs.100* who want to maximize the expected utility of terminal wealth.

It is difficult to calculate optimal $\gamma(i,x)$values numerically for an arbitrary distribution using (6.3). The approach here is similar to Çanakoğlu and Özekici (2009, 2010) i.e., is to use Taylor series expansion of the utility function around the expected value $\overline{W} = E[W]$ of the terminal wealth $W = X_T$. Jondeau and Rockinger (2006) gives a detailed discussion on the benefits, advantages and disadvantages of using Taylor series expansion in optimal portfolio allocation. In particular, they give a convincing argument for using the first 4 moments in the approximation. From the data it is recognized that the return distributions have very small skew-ness and kurtosis, so it was decided to use the first four moments. Taylor series expansion is

$$U(W) = \sum_{j=0}^{+\infty} U^{(j)}(\bar{W}) \frac{(W-\bar{W})^j}{j!}$$

where$U^{(j)}(\bar{W})$ is the j^{th} derivative of the utility function at $\bar{W}$. Taking expectations the above equation can be written as,

$$E[U(W)] = U(\bar{W}) + \frac{1}{2!}U^2(\bar{W})\mu_p^2 + \frac{1}{3!}U^3(\bar{W})\mu_p^3 + \frac{1}{4!}U^4(\bar{W})\mu_p^4 + E[R_4(W,\bar{W})] \quad (7.1)$$

where$R_4(W,\bar{W})$ is the remainder for the first four moments and μ_p^n is the nth moment of the portfolio defined as

$$\mu_p^2 = E[(W-\bar{W})^n]$$

Using the definitions in Jondeau and Rockinger(2006) for any market state, the second moment can be expressed as

$$\mu_p^2 = \gamma' M_2 \gamma$$

where$M_2 = \boldsymbol{\Sigma}(.)$ is the covariance matrix. Similarly

$$\mu_p^3 = \gamma' M_3 (\gamma \otimes \gamma)$$

where $\otimes$ is the Kronecker product, and M_3 is the 3 x 9 co-skewness matrix defined as

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right]$$

with

$$s_{ijk} = E[(R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)]$$

for *i*, *j*,*k*=*1*,*2*,*3*. Finally,

$$\mu_p^4 = \gamma' M_4 (\gamma \otimes \gamma \otimes \gamma)$$

where M_4 is the 3 x 27 co-kurtosis matrix with elements

$$k_{ijkl} = E[(R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)(R_l-\mu_l)]$$

for *i,j,k,l=1,2,3*.

For PLEC utility function

$$U(x) = -e^{\beta x}(x-\eta)^{\alpha}$$

Equation (7.1) can be written as,

$$E[U(\overline{W})] \cong \Big[-e^{\beta W}(W-\eta)^{\alpha} - \frac{1}{2}e^{\beta W}(W-\eta)^{\alpha-2}[[\alpha+\beta(W-\eta)]^2-\alpha]\mu_p^2 - \frac{1}{6}e^{\beta W}(W-\eta)^{\alpha-3}[3(W-\eta)(-\alpha\beta+\beta\alpha^2)+3(W-\eta)^2\beta^2\alpha + (W-\eta)^3\beta^3 - 3\alpha^2+2\alpha]\mu_p^3 - \frac{1}{24}e^{\beta W}(W-\eta)^{\alpha-4}[(W-\eta)^4\beta^3+(W-\eta)^3\alpha\beta^3+3\beta^3\alpha) + 3(W-\eta)^2(\beta^2\alpha^2-\beta^2\alpha-\alpha\beta+\beta\alpha^2)+(W-\eta)(6\alpha\beta + 3\beta\alpha^3-9\beta\alpha^2-3\alpha^2+2\alpha)]\mu_p^4\Big]$$

Based on Theorem 6.1, take $W = xr_f(R^{e'}\gamma(i,x)+1)$, in the Taylor series expansion (7.1). When the data was checked, both covariances and expected excess returns are of the order 0.01. So for$W = xr_f(R^{e'}\gamma(i,x)+1)$, it can be supposed that$0 < W < 2\overline{W}$. Then based on the study by Loistl (1976), it can be said that the series converges most of the time. The Taylor series expansion is given in the Appendix.

Consider the PLEC utility where 3 types of investors are studied. Cases where the investor has his risk preferences as IARA$(\alpha = 1, \beta = -5)$,as CARA$(\alpha = 0, \beta = -1)$and as DARA $(\alpha = 1, \beta = 0)$ are considered here. Let η=99 and C(i) and K(i)=1.

Illustration 1:

Consider an investor having his risk preference as IARA with parameters$(\alpha = 1, \beta = -5)$. Now substitute these parameter values in the Taylor series expansion of (7.1) (given in Appendix) and take the gradient with respect to $\gamma(i,x)$ and set it to equal to zero. Thus the first order condition from which the optimal $\gamma(i,x)$values are determined numerically using R program for each market state is obtained. The computational time was less than a half a minute on a laptop with 2.1 GHz process.

The optimal $\gamma$for the rest of the periods can be obtained similarly by using the normalized values of η and β for different periods. The corresponding optimal investment amount is given as,

Table 2.Optimal amounts to be invested in each of the assets for each of the states

State / Asset	1	2	3	4
Reliance	6.489	-1.188	2.193	-1.958
ACC	2.881	-0.087	0.797	-2.102
Tata Steel	2.105	-0.133	-0.053	-2.107
Risk free asset	88.526	101.409	97.062	106.167

Notice that since the investor is risk-averse, large proportion of the wealth is invested in the risk free asset. The negative signs indicate short selling. Also the result suggests that one need not short sell if the market is in state 1 where in the market has moved up twice. This observation is quite logical for any investor in the real world. Similarly when the market has gone down previously a risk averse investor may short sell all the risky assets and invest more in the risk free asset. One can clearly observe these points from Table 2.

Illustration 2:

Consider an investor having his risk preference asCARA with parameters$(\alpha = 0, \beta = -1)$. The procedure similar to as done in illustration 1 is applied. The corresponding optimal investment amount is given as,

Table 3.Optimal amounts to be invested in each of the assets for each of the states

State / Asset	1	2	3	4
Reliance	18.792	-5.861	10.469	-7.677
ACC	8.342	-0.431	3.806	-8.242
Tata Steel	6.096	-0.659	-0.253	-8.261
Risk free asset	66.771	106.951	85.979	124.181

The observations regarding the amounts to be invested are similar to that of Illustration 1.

Illustration 3:

Consider an investor having his risk preference as DARA with parameters$(\alpha = 1, \beta = 0)$. The procedure similar to as done in illustration 1 is applied. The corresponding optimal investment amount is given as,

Table 4.Optimal amounts to be invested in each of the assets in each of the state

State / Asset	1	2	3	4
Reliance	778.378	-291.500	512.189	-344.777
ACC	345.814	-21.454	186.208	-370.158
Tata Steel	252.813	-32.760	-12.386	-370.996
Risk free asset	-1277.005	445.714	-586.011	1185.932

Notice that the policy suggests a large amount to be de-invested in the risk free asset for state 1 and 3 where in the market has moved up in the last period. This happens because the investor considered here is of type DARA. Thus as he is willing to take more risk as the market moves up, the investor may put a larger proportion of his wealth in the risky assets and vice-versa in state 2 and 4.

Illustration 4:

In this illustration we have considered a situation where the investor has his risk preference to be changing in each state to IARA, CARA and DARA with parameters$(\alpha_i, \beta_i) = \{(1,5), (0,-1), (0,-0.5), (-1,0)\}$respectively. The procedure similar to as done in illustration 1is applied.The corresponding optimal investment amount is given as,

Table 5.Optimal amounts to be invested in each of the assets in each of the state

State / Asset	1	2	3	4
Reliance	6.489	-5.861	10.584	-344.041
ACC	2.881	-0.431	3.848	-369.364
Tata Steel	2.105	-0.659	-0.256	-370.249
Risk free asset	88.526	106.951	85.824	1183.654

This illustration shows how one can consider different risk preferences of investors based on each state to obtain the optimum policy. In reality an investor may change his risk preference based on the state of the market. Hence it becomes an important part of the study that this scenario is also considered. This type of study is possible only through the PLEC utility as it incorporates all these types of risk preference structures. Hence one is free from any bias in the optimum policy that could occur because of the change in form of the utility.

Implications of an ignorant investment strategy

Suppose an investor does not use any information about the variation of the states of the market. He just knows what type of risk preference he possesses. The solution for such an investor obtained without dynamic programming, is a suboptimal solution compared to the one obtained through dynamic programming. This can be shown by obtaining the optimal policy for a single period by solving the first order condition equation (5.3) and utility of an IARA preference type investor.

If we apply this policy for Rs. 1000 investment and compare it with the dynamic problem solution in each state for the same utility function and initial wealth the following loss in rupees will occur:

Table 6. Loss of wealth in rupees for Rs. 1000 invested using the policy obtained from first order condition rather than dynamic programming

State	1	2	3	4
Loss	1.03	1.05	1.05	1070.09

Thus Table 6 suggests that one may incur lesser profit if dynamic programming technique is not used to obtain the optimal policy for a multiple period horizon problem.

8. Conclusions

PLEC utility is recommended here for studying the investor preferences and for obtaining an optimal policy to the portfolio allocation problem of the various types of risk-averse investors. This makes the optimum policy obtained to be free from any bias due to change in the form of the utility function considered. A direct solution to a single period problem is obtained based on this utility. Price variation is modelled through a Markov chain. The optimal decision in a multi-period setting is obtained using dynamic programming.

In both the investment horizons, it is noticed that the decision depends on the wealth invested and the state of the market. In the multi-period setting the optimal policy depends on the period of investment as well. Thus it is noticed that using the PLEC utility provides us a non-myopic policy which is otherwise not possible when any other utility function is used.

Four illustrations are shown which discuss about three different kinds of investors. The investors are represented through different PLEC utilities. Optimal policies can be compared for each investor type without bothering about the bias that could have occurred by the use of different forms of utility functions. This type of study is possible uniquely through PLEC utility because of its wide ranged behaviour regarding risk-averse preferences. It is seen that not bothering about the market's stochastic state may lead to lesser profit for the investor. Hence we recommend the use of PLEC utility function to represent the various investors in the market. So also based on this utility, we suggest using the method described here to obtain the optimal portfolio solutions for the investors under consideration.

Appendix

Based on Theorem 6.1, take $W = xr_f(R^{e'}\gamma(i,x) + 1)$, in the Taylor series expansion (7.1). Therefore the Taylor series expansion can be used as,

$$
\begin{aligned}
&E\left[U\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)\right)\right] \\
&\qquad \cong \Bigg[-e^{\beta\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)}\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta\right)^{\alpha} \\
&\qquad -\frac{1}{2}e^{\beta\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)}\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)\right. \\
&\qquad \left.-\eta\right)^{\alpha-2}\left[\left[\alpha+\beta\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta\right)\right]^2-\alpha\right]\mu_p^2 \\
&\qquad -\frac{1}{6}e^{\beta\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)}\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)\right. \\
&\qquad \left.-\eta\right)^{\alpha-3}\Bigg[3\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta\right)(-\alpha\beta+\beta\alpha^2) \\
&\qquad +3\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta\right)^2\beta^2\alpha \\
&\qquad +\left(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta\right)^3\beta^3-3\alpha^2+2\alpha\Bigg]\mu_p^3\Bigg]
\end{aligned}
$$

$$-\frac{1}{24}e^{\beta\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)}(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta)^{\alpha-4}\Big[(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta)^4\beta^3+(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta)^3\alpha\beta^3+3\beta^3\alpha\Big)+3(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta)^2(\beta^2\alpha^2-\beta^2\alpha-\alpha\beta+\beta\alpha^2)+(\left((xr_f-\eta)R^{e'}\gamma(i,x)+xr_f\right)-\eta)(6\alpha\beta+3\beta\alpha^3-9\beta\alpha^2-3\alpha^2+2\alpha)]\mu_p^4$$

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